

An Alternative Approach to Unification of Gauge and Geometric Interactions

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Abstract

The algebra of the generators for infinitesimal transformations of the $\Gamma = \frac{1}{2}$ representation of causal spinor fields (Dirac fields) *explicitly* constructs the Minkowski metric *within* the internal group space as a consequence of non-vanishing commutation relations between generators that carry a single space-time index. This representation is a subgroup of the set of all of the generators that transform under the group $GL(4)$. The sixteen hermitian generators of $GL(4)$ include the three angular momentum spin matrices, a matrix proportional to the Dirac matrix γ^0 , and 12 additional matrices that have the same number of degrees of freedom as $SU(3) \times SU(2) \times U(1)$. In this paper, the construction of linearly independent internal $SU(3)$ and $SU(2)$ local symmetry groups for the causal spinor fields is demonstrated to necessarily involve the CKM mixing of three sets of the $SU(3)$ eigenstates as related to the $SU(2)$ eigenstates.

1 Introduction

Perhaps the most fundamental characteristic of the standard model implementation of fundamental micro-physical interactions is their relation to the gauge symmetry product group $SU(3) \times SU(2) \times U(1)$. Local unitary gauge transformations generally refer to an invariance of physical measurables under transformations of the particle fields of the form $\Psi(x) \rightarrow \tilde{\Psi}(x) = \mathbf{U}_{(a)}(\underline{\alpha}(x))\Psi(x)$, where $\mathbf{U}_{(a)}$ is a unitary representation of a group of transformations $\mathcal{G}_{(a)}$, and the group parameters $\underline{\alpha} = \underline{\alpha}(x)$ can take on an arbitrary functional dependency on the space-time location x . In order to implement this symmetry, vector gauge potentials $A_\mu^s(x)$ are introduced using “minimal coupling”, with the requirement that they transform under the gauge transformation according to $\mathbf{U}_{(a)}^{-1}(\underline{\alpha}) \frac{\hbar}{i} \frac{\partial}{\partial x^\mu} \mathbf{U}_{(a)}(\underline{\alpha}) = \frac{q_a}{c} \sum_s A_\mu^s \mathbf{U}_{(a)}^{-1}(\underline{\alpha}) \mathbf{G}_s^{(a)} \mathbf{U}_{(a)}(\underline{\alpha})$, where $\mathbf{G}_s^{(a)}$ is the generator of infinitesimal translations of group parameter α^s . The parameter q_a is the charge coupling the gauge fields $A_\mu^s(x)$ to particle currents, making these vector gauge potentials the carriers of micro-physical interactions between the particles.

General relativity has been consistently successful in describing the macro-physical phenomena of gravitation, as well as the gravitation of coherent quantum states[1, 2, 3]. The theory formulates equations of motion for systems on curvilinear space-times, where the local curvature is generated by the local energy-momentum density. All measurables can be derived from a local and symmetric space-time metric function $g_{\mu\nu}(x)$ that through the principle of equivalence can be transformed into a locally flat Minkowski metric form $\eta_{\alpha\beta} = \frac{\partial x^\mu}{\partial \xi^\alpha} g_{\mu\nu} \frac{\partial x^\nu}{\partial \xi^\beta}$ on coordinates $\xi(x)$. The metric $\eta_{\alpha\beta}$ cannot be explicitly generated by the

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Lorentz group algebra through a Casimir-like construction, since there are no non-abelian generators in that algebra that carry a single space-time index.

The Dirac equation[4] for spin $\frac{1}{2}$ fermions utilizes a matrix algebra to construct a relativistic equation for particle wave functions ψ that is linear in the quantum operators for 4-momentum, generates a positive semi-definite probability density $\psi^\dagger\psi$, and maintains the expected non-relativistic correspondence with the Schrodinger equation[5]. In particular, the evolution dynamics describing equations that are linear in energy-momentum operators is mathematically straightforward[6]. The development of Dirac's equation required the introduction of 4×4 matrices that each carry a single space-time index.

The Dirac formulation can be extended by developing general operators $\hat{\Gamma}^\mu$ that require the form $\hat{\Gamma}^\mu \hat{P}_\mu$ to be a Lorentz scalar with an eigenvalue linear in the particle mass. The field equation that results takes the form

$$\mathbf{\Gamma}^\mu \cdot \left(\frac{\hbar}{i} \frac{\partial}{\partial x^\mu} - \frac{q_a}{c} \sum_s A_\mu^s \mathbf{G}_s \right) \hat{\Psi}_\gamma^{(\Gamma)}(\vec{x}) = -\gamma mc \hat{\Psi}_\gamma^{(\Gamma)}(\vec{x}), \quad (1.1)$$

where m is positive semi-definite for all particle types (both particles and antiparticles), and the $\mathbf{\Gamma}^\mu$ are finite dimensional matrix representations of the operators $\hat{\Gamma}^\mu$. In this equation, γ is the eigenvalue of the operator $\hat{\Gamma}^0$ on the standard (rest) state particle spinor, and this eigenvalue takes differing signs for particles vs. antiparticles. Using this formulation, energies are always non-negative. This form is seen to be invariant under local gauge transformations of the form previously introduced. The Dirac matrices result from the $\Gamma = \frac{1}{2}$ finite dimensional representation of the operators $\hat{\Gamma}^\mu$. A representation for the $\Gamma = 1$ matrices can be found in Appendix D.2.1 of reference [6]. In what follows, the group structure of the algebra will be briefly discussed, and internal symmetries of the causal fields $\hat{\Psi}_\gamma^{(\frac{1}{2})}$ will be explored.

2 Properties of a Closed Group Inclusive of Γ^μ

The closed algebra that includes the operators $\hat{\Gamma}^\mu$ add the following commutation relations to those of the Lorentz group:

$$\begin{aligned} [\Gamma^0, \Gamma^k] &= \frac{i}{\hbar} K_k, & [\Gamma^0, J_k] &= 0, & [\Gamma^0, K_k] &= -i\hbar \Gamma^k, \\ [\Gamma^j, \Gamma^k] &= -\frac{i}{\hbar} \epsilon_{jkm} J_m, & [\Gamma^j, J_k] &= i\hbar \epsilon_{jkm} \Gamma^m, & [\Gamma^j, K_k] &= -i\hbar \delta_{jk} \Gamma^0. \end{aligned} \quad (2.2)$$

Eigenvalues Γ of the Casimir operator $C_\Gamma = \frac{1}{6} ((\underline{J} \cdot \underline{J} - \underline{K} \cdot \underline{K})/\hbar^2 + \Gamma^0 \Gamma^0 - \underline{J} \cdot \underline{J})$ will label the irreducible representations of this group. Additional labels can be attributed to the eigenvalues of the mutually commuting operators C_Γ , Γ^0 , J^2 , and J_3 , given by $\frac{2\Gamma(\Gamma+2)}{6}$, γ , $J(J+1)\hbar^2$, and $M\hbar$, respectively.

Finite dimensional representations of the algebra have dimension $N_\Gamma = \frac{1}{3}(\Gamma+1)(2\Gamma+1)(2\Gamma+3)$, and general representation eigenstates $\chi_{\gamma,M}^{\Gamma,J}$ can be constructed using the spinor form

$$\chi_{\gamma,M}^{\Gamma,J} = \sqrt{\frac{(J-M)!}{(2J)!(J+M)!}} [x-y]^{\Gamma-J} \chi_+^{(+M+\gamma)} \chi_+^{(-M-\gamma)} \left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right]^{J+M} x^{J-\gamma} y^{J+\gamma} \bigg|_{\substack{x = \chi_+^{(+)} \chi_-^{(-)} \\ y = \chi_-^{(+)} \chi_+^{(-)}}}, \quad (2.3)$$

where the $\chi_{\pm}^{(\pm)}$ are $\Gamma = \frac{1}{2}$ representation spinors with $(\gamma) = (\pm\frac{1}{2})$ and $M = \pm\frac{1}{2}$. The signatures (half-integral vs integral) of Γ , J , γ , and M are necessarily the same, with $0 \leq J \leq \Gamma$, $-J \leq \gamma \leq J$, and $-J \leq M \leq J$. The spinor forms of the operators satisfying the commutation relations (2.2) can be found in [6]. The matrices corresponding to $\Gamma = \frac{1}{2}$ have dimensionality $N_{\frac{1}{2}} = 4$, with $\mathbf{\Gamma}^0 = \frac{1}{2}\gamma_{Dirac}^0$ and \mathbf{J}_3 both diagonal. This representation forms a subgroup of $GL(4)$.

2.1 Development of a group metric on space-time indexes

For a general group algebra $[\hat{G}_r, \hat{G}_s] = -i \sum_m (c_s)_r^m \hat{G}_m$ with appropriately non-vanishing structure constants $c_{sr}^m = -c_{rs}^m$, the Jacobi identity defines a representation in terms of the structure constants which can be used to construct a group metric η_{ab} [7],

$$\eta_{ab} \equiv \sum_{s r} (c_a)_r^s (c_b)_s^r, \quad \eta^{ab} \equiv ((\eta)^{-1})_{ab}. \quad (2.4)$$

This *group* metric on the generators defines invariants on products of group generators, such as the Casimir operator $\hat{C}_g \equiv \sum_{rs} \eta^{G_r G_s} \hat{G}_r \hat{G}_s$. The non-commuting operators $\hat{\Gamma}^\mu$ that carry a space-time index conjugate to the 4-momentum define a group metric of Lorentz sub-group invariants given by

$$\eta^{\Gamma^\mu \Gamma^\nu} = -\frac{1}{6} \eta_{\mu\nu} \quad (2.5)$$

where $\eta_{\mu\nu}$ is the usual Minkowski metric. Thus, the Minkowski metric defining invariant products of group generators is *explicitly* generated within this closed algebra, beyond the Lorentz invariance *implicit* in Lorentz transformations. This *group theoretic* metric can be used to develop Lorentz invariants of *any* operators carrying the group indexes of Γ^μ . Once space-time translations are incorporated, since the generators of space-time translations P_μ transform as covariant 4-vectors under arbitrary coordinate transformations (of which group transformations are a special subset[8]), this group structure is explicitly tied to curvilinear space-time dynamics expressing the principle of equivalence.

3 Construction of Local Invariance Groups

In this section, the invariance groups of $\Gamma = \frac{1}{2}$ representation eigenstates will be explored.

3.1 A complete set of linearly independent hermitian generators in $GL(4)$

As previously mentioned, the $\Gamma = \frac{1}{2}$ representation is a subgroup of $GL(4)$. There are 16 hermitian matrices in $GL(4)$, including that associated with the $U(1)$ transformation proportional to the identity. The subgroup connected to particle representations already includes the 4 hermitian generators Γ^0 and the angular momenta J_k . This leaves an additional 11 matrices, which will be constructed as $\mathbf{T}_j = i \Gamma^j$ and $\mathbf{T}_{j+3} = i \mathbf{K}_j$ for $j = 1 \rightarrow 3$, two matrices given by

$$\mathbf{T}_7 = \frac{i}{2} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}, \quad \mathbf{T}_8 = \frac{1}{2} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \quad (3.6)$$

where $\mathbf{1}$ and $\mathbf{0}$ are the 2×2 identity and zero matrices respectively, and a final set of three generators \mathbf{T}_9 , \mathbf{T}_{10} , and \mathbf{T}_{11} forming a closed representation of $SU(2)$ on the lower components given by

$$\mathbf{T}_{j+8} = \frac{1}{2} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sigma_j \end{pmatrix}, \quad (3.7)$$

where the σ_j are Pauli spin matrices.

It is intriguing that the number of additional hermitian generators in $GL(4)$ is precisely the same as the number of hermitian generators in $SU(3) \times SU(2) \times U(1)$. However, the set of 8 Hermitian generators \mathbf{T}_s for $s : 1 \rightarrow 8$ do not form a closed algebra.

3.2 Transformation properties of causal spinor fields

Causal spinor fields $\hat{\Psi}(x)$ either commute or anti-commute $[\hat{\Psi}(x), \hat{\Psi}(y)]_{\mp} = 0$ for space-like separations of the space-time coordinates $(\vec{y} - \vec{x})$ of those fields according to whether the spin is integral or half-integral. Thus, microscopic causality compels a well defined *local* relationship between components of spinor fields in configuration space. The form of a causal linear spinor field that has the expected properties under parity, time reversal, and charge conjugation is given by[6]

$$\begin{aligned} \hat{\Psi}_{\gamma}^{(\Gamma)}(\vec{x}) \equiv \sum_{J, s_z} \int \frac{mc^2 d^3 p}{\epsilon(\mathbf{p})} \left[\frac{e^{\frac{i}{\hbar}(\mathbf{p} \cdot \mathbf{x} - \epsilon(\mathbf{p})t)}}{(2\pi\hbar)^{3/2}} \mathbf{u}_{\gamma}^{(\Gamma)}(\vec{p}, m, J, s_z) \hat{a}_{\gamma}^{(\Gamma)}(\vec{p}, m, J, s_z) + \right. \\ \left. (-)^{J+s_z} \frac{e^{-\frac{i}{\hbar}(\mathbf{p} \cdot \mathbf{x} - \epsilon(\mathbf{p})t)}}{(2\pi\hbar)^{3/2}} \mathbf{u}_{-\gamma}^{(\Gamma)}(\vec{p}, m, J, -s_z) \hat{a}_{-\gamma}^{(\Gamma)\dagger}(\vec{p}, m, J, s_z) \right], \end{aligned} \quad (3.8)$$

where the normalization has been chosen to have non-relativistic correspondence, $\frac{mc^2 d^3 p}{\epsilon(\mathbf{p})} \rightarrow d^3 p$ for $p \ll mc$, and s_z specifies the z-component of internal angular momentum to avoid confusion with the mass m . In this expression, the $\hat{a}_{\gamma}^{(\Gamma)}(\vec{p}, m, J, s_z)$ are annihilation operators of particle states of the given quantum numbers, $\epsilon(\mathbf{p}) = \sqrt{|\mathbf{p}|^2 c^2 + m^2 c^4}$, and the spinors satisfy $\Gamma^{\mu} p_{\mu} \mathbf{u}_{\gamma}^{(\Gamma)}(\vec{p}, m, J, s_z) = -\gamma mc \mathbf{u}_{\gamma}^{(\Gamma)}(\vec{p}, m, J, s_z)$. In this paper, the focus will be on the $\Gamma = \frac{1}{2}$ representation, for which the $\mathbf{u}_{\gamma}^{(\frac{1}{2})}$ have 4 spinor components. For a more complete treatment of the algebra, symmetries, and causality properties of causal spinor fields, the reader is invited to examine sections 4.3 and 4.4 in reference [6].

Since the spinor components of a causal 4-spinor field $\Psi^{(\frac{1}{2})}$ maintain local spatial relationships described in (3.8), a particular local Euclidean rotation $\mathbf{R}_E(x)$ will be constructed to define a spinor with a single component. If the components of the causal 4-spinor are written in the form

$$\Psi(x) = \begin{pmatrix} \phi_1(x) e^{i\omega_1(x)} \\ \phi_2(x) e^{i\omega_2(x)} \\ \phi_3(x) e^{i\omega_3(x)} \\ \phi_4(x) e^{i\omega_4(x)} \end{pmatrix} \equiv \begin{pmatrix} \phi_1(x) e^{i\omega_1(x)} \\ -\phi_1(x) \tan(\zeta_{12}) \sec(\zeta_{13}) \sec(\zeta_{14}) e^{i\omega_2(x)} \\ -\phi_1(x) \tan(\zeta_{13}) \sec(\zeta_{14}) e^{i\omega_3(x)} \\ -\phi_1(x) \tan(\zeta_{14}) e^{i\omega_4(x)} \end{pmatrix}, \quad (3.9)$$

then the causal spinor can be expressed by the action of a Euclidean rotation on a single component spinor

as follows:

$$\begin{aligned} \mathbf{R}_{14} &= \begin{pmatrix} \cos(\zeta_{14}) & 0 & 0 & \sin(\zeta_{14})e^{i\omega_{14}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin(\zeta_{14})e^{-i\omega_{14}} & 0 & 0 & \cos(\zeta_{14}) \end{pmatrix}, \quad \mathbf{R}_{13} = \begin{pmatrix} \cos(\zeta_{13}) & 0 & \sin(\zeta_{13})e^{i\omega_{13}} & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\zeta_{13})e^{-i\omega_{13}} & 0 & \cos(\zeta_{13}) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ \mathbf{R}_{12} &= \begin{pmatrix} \cos(\zeta_{12}) & \sin(\zeta_{12})e^{i\omega_{12}} & 0 & 0 \\ -\sin(\zeta_{12})e^{-i\omega_{12}} & \cos(\zeta_{12}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \omega_{1s} \equiv \omega_1 - \omega_s, \quad \mathbf{R}_E \equiv \mathbf{R}_{14}\mathbf{R}_{13}\mathbf{R}_{12}, \quad \Psi(x) = \mathbf{R}_E(x)\bar{\Phi}(x). \end{aligned} \quad (3.10)$$

The single component spinor takes the form

$$\bar{\Phi}(x) = \begin{pmatrix} \sqrt{(\phi_1(x))^2 + (\phi_2(x))^2 + (\phi_3(x))^2 + (\phi_4(x))^2} e^{i\omega_1(x)} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{R}_E^{-1}(x)\Psi(x). \quad (3.11)$$

It is important to note that this transformation must be made uniquely at each location x . This imposes a *local* nature to any internal symmetries that utilize this construction.

3.3 SU(2) and SU(3) invariance transformations on $\bar{\Phi}(x)$

The previously defined generators $\tau_j \equiv \mathbf{T}_{j+8}$ with $j : 1 \rightarrow 3$ form a closed matrix group algebra $\mathbf{M}^{(2)}$ that transform components in the subspace $\mathbf{S}^{(2)}$:

$$\mathbf{M}^{(2)}(\underline{\theta}) = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^{(2)}(\underline{\theta}) \end{pmatrix} = e^{i \sum \theta^s \tau_s}. \quad (3.12)$$

In this expression, the matrix $\mathbf{S}^{(2)}$ is a unitary unimodular transformation matrix in SU(2), and $\mathbf{1}$ is the 2×2 identity matrix. Since $\tau_j \bar{\Phi}(x) = 0$, the field $\bar{\Phi}(x)$ from Eq. 3.11 is invariant under transformations involving $\mathbf{M}^{(2)}$. The generators τ_j share this basis simultaneously with \mathbf{I}^0 and \mathbf{J}_3 .

As was previously mentioned, the remaining hermitian generators in this basis $\{\mathbf{T}_1, \dots, \mathbf{T}_8\}$ do not form a closed algebra. However, a basis of SU(3) states *can* be constructed through mixing of the hermitian generators in this basis. This mixing between precisely three sets of the basis states will next be demonstrated.

Consider the set of SU(3) transformations of the form

$$\mathbf{M}^{(3)}(\underline{\alpha}) = \begin{pmatrix} 1 & \bar{\mathbf{0}}^T \\ \bar{\mathbf{0}} & \mathbf{S}^{(3)}(\underline{\alpha}) \end{pmatrix} = e^{i \sum \alpha^b \mathbf{t}_b}, \quad (3.13)$$

where $\mathbf{S}^{(3)}$ is a unitary unimodular transformation matrix in SU(3), and $\bar{\mathbf{0}}$ is a 1×3 zero vector. The set of eight generators $\{\mathbf{t}_1, \dots, \mathbf{t}_8\}$ form a closed algebra of SU(3). Since $\mathbf{t}_b \bar{\Phi}(x) = 0$, the field $\bar{\Phi}(x)$ is invariant under transformations involving $\mathbf{M}^{(3)}$. The SU(3) eigenstates will be defined using this basis. A set of SU(3) generators (including 2 diagonal generators) can be found in [9].

The construction of a representation of $SU(3)$ that is inclusive of the hermitian generators $\{\Gamma^0, \mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3\}$ in $GL(4)$ necessarily involves the mixing of the three basis states of the transformation (3.13). A convenient mechanism for mixing three eigenstates is provided through general CKM matrices[10, 11] embedded within $GL(4)$. The particular choice for mixing will be the set of all transformations on the 3×3 subspace that leaves the second spin component of $\bar{\Phi}(x)$ invariant as demonstrated below:

$$\mathbf{U}_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\theta_{23}) & \sin(\theta_{23}) \\ 0 & 0 & -\sin(\theta_{23}) & \cos(\theta_{23}) \end{pmatrix}, \quad \mathbf{U}_{31} = \begin{pmatrix} \cos(\theta_{31}) & 0 & 0 & \sin(\theta_{31})e^{-i\delta_{31}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin(\theta_{31})e^{i\delta_{31}} & 0 & 0 & \cos(\theta_{31}) \end{pmatrix},$$

$$\mathbf{U}_{12} = \begin{pmatrix} \cos(\theta_{12}) & 0 & \sin(\theta_{12}) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta_{12}) & 0 & \cos(\theta_{12}) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{U}_{CKM} \equiv \mathbf{U}_{23}\mathbf{U}_{31}\mathbf{U}_{12}. \quad (3.14)$$

Most CKM transformations of this type on the generators $\tilde{\mathbf{t}}_s \equiv \mathbf{U}_{CKM}\mathbf{t}_s\mathbf{U}_{CKM}^{-1}$ will produce a set of transformed generators $\tilde{\mathbf{t}}_s$ that both satisfy the algebra of $SU(3)$ as well as can be included among a set of 11 linearly independent traceless hermitian generators that complete the set $\{\Gamma^0, \mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3\}$.

For clarity, consider an example CKM transformation from the $SU(3)$ eigenbasis to that including the generators $\{\mathbf{T}_1, \dots, \mathbf{T}_{11}\}$ parameterized using data consistent with known mixing between the generations in quark phenomenology, $(\theta_{12} \rightarrow 0.227, \theta_{23} \rightarrow 0.446, \theta_{31} \rightarrow 0.0035, \delta_{31} \rightarrow 1.96)$:

$$\mathbf{U}_{CKM} = \begin{pmatrix} 0.974 & 0 & 0.225 & -0.0013 - 0.0032i \\ 0 & 1 & 0 & 0 \\ -0.225 + 0.00014i & 0 & 0.973 + 0.000033i & -0.0446 \\ -0.00875 - 0.00316i & 0 & 0.0437 - 0.00073i & 0.999 \end{pmatrix}. \quad (3.15)$$

The set of 15 linearly independent hermitian generators $\{\Gamma^0, \mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3, \mathbf{T}_1, \dots, \mathbf{T}_{11}\}$ can be directly decomposed in terms of an alternative set of 15 linearly independent hermitian generators given by

$$\{\Gamma^0, \mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3, \tilde{\mathbf{t}}_1, \dots, \tilde{\mathbf{t}}_8, \mathbf{\Delta}_1 \equiv (\mathbf{T}_1 - \mathbf{T}_5)/2, \mathbf{\Delta}_2 \equiv (\mathbf{T}_2 + \mathbf{T}_4)/2, \mathbf{\Delta}_3 \equiv (\mathbf{T}_3 + \mathbf{T}_7)/2\}.$$

Although the set of matrices $\{\tilde{\mathbf{t}}_1, \dots, \tilde{\mathbf{t}}_8\}$ now generate a closed $SU(3)$ algebra in this basis, the remaining generators $\mathbf{\Delta}_1, \mathbf{\Delta}_2$, and $\mathbf{\Delta}_3$ *do not* form a closed algebra. The transformed $SU(3)$ generators $\tilde{\mathbf{t}}_b$ are decomposed in (3.16).

	Γ^0	J_1	J_2	J_3	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}
$\tilde{\mathbf{t}}_1$	-.295	0	0	-.295	-.071	112	-1.58	112	.071	-5.65	-1.58	-5.65	970	0	-86.8
$\tilde{\mathbf{t}}_2$	-.730	0	0	-.730	112	-.071	-4.38	-.071	-112	-1.58	-4.38	-1.58	-.065	-974	.003
$\tilde{\mathbf{t}}_3$	-25.3	0	0	-25.3	-1.7	-5.58	.076	-5.58	1.7	-109	.076	-109	-87.1	.711	-945
$\tilde{\mathbf{t}}_4$	0	-1.31	3.24	0	0	22.3	0	-22.3	0	-499	0	499	1.31	-3.24	0
$\tilde{\mathbf{t}}_5$	0	-3.24	-1.31	0	-22.3	0	-500	0	-22.3	0	500	0	3.24	1.31	0
$\tilde{\mathbf{t}}_6$	0	225	0	0	-.016	-487	-.365	487	-.016	-21.9	.365	21.9	-225	0	0
$\tilde{\mathbf{t}}_7$	0	0	225	0	487	-.016	-21.9	.016	487	.365	21.9	-.365	0	-225	0
$\tilde{\mathbf{t}}_8$	-975	0	0	1025	-1.54	4.26	.069	4.26	1.54	110	.069	110	-1.97	-.711	-1051

(3.16)

For each of the transformed generators $\tilde{\mathbf{t}}_b = \sum_j k_b^j \mathbf{G}_j$, the coefficients k_b^j are to be obtained by multiplying the number in the column under the previously defined generator \mathbf{G}_j by 10^{-3} in (3.16). It is clear that

the set of independent generators $\{\Gamma^0, \mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3, \tilde{\mathbf{t}}_1, \dots, \tilde{\mathbf{t}}_8, \Delta_1, \Delta_2, \Delta_3\}$ that includes a closed SU(3) sub-algebra $\{\tilde{\mathbf{t}}_1, \dots, \tilde{\mathbf{t}}_8\}$ necessarily mixes in an invertible manner the basis $\{\Gamma^0, \mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3, \mathbf{T}_1, \dots, \mathbf{T}_{11}\}$ that includes an internal closed SU(2) sub-algebra $\{\mathbf{T}_9, \mathbf{T}_{10}, \mathbf{T}_{11}\}$ on $\bar{\Phi}(x)$. The independent group of SU(3) transformations on the 4-spinors generated by $\{\tilde{\mathbf{t}}_1, \dots, \tilde{\mathbf{t}}_8\}$ will be denoted $\mathbf{U}^{(3)}(\underline{\alpha}(x))$, so that

$$\mathbf{M}^{(3)} \equiv \mathbf{U}_{CKM}^{-1} \mathbf{U}^{(3)} \mathbf{U}_{CKM}, \quad (3.17)$$

where $\mathbf{M}^{(3)}$ is of the form defined in Eq. 3.13.

3.4 Local gauge symmetries of spinor fields

A local internal SU(3) symmetry on the causal spinor field $\hat{\Psi}_\gamma^{(\frac{1}{2})}(\vec{x})$ from (3.8) can be constructed using the internal SU(3) symmetry transformation $\mathbf{M}^{(3)}$ on $\bar{\Phi}(x)$ from (3.17), along with the relationship of the transformed spinor $\bar{\Phi}(x)$ to the general causal spinor field expressed in (3.11), via

$$\mathcal{U}^{(3)}(x) = \mathbf{R}_E(x) \mathbf{M}^{(3)}(\underline{\alpha}(x)) \mathbf{R}_E^{-1}(x) = \mathbf{R}_E(x) \mathbf{U}_{CKM}^{-1} \mathbf{U}^{(3)}(\underline{\alpha}(x)) \mathbf{U}_{CKM} \mathbf{R}_E^{-1}(x), \quad (3.18)$$

where $\alpha^s(x)$ are the eight local group parameters of SU(3). Similarly, the transformation $\mathbf{M}^{(2)}$ from (3.12) on $\bar{\Phi}(x)$ defines a local internal SU(2) symmetry on the causal spinor field $\hat{\Psi}_\gamma^{(\frac{1}{2})}(\vec{x})$ under transformations

$$\mathcal{U}^{(2)}(x) = \mathbf{R}_E(x) \mathbf{M}^{(2)}(\underline{\theta}(x)) \mathbf{R}_E^{-1}(x), \quad (3.19)$$

where $\theta^j(x)$ are the three local group parameters of SU(2). Although both (3.18) and (3.19) are local internal symmetries on the causal spinor field $\mathcal{U}(x)\Psi(x) = \Psi(x)$, the eigenbases of the internal SU(2) and SU(3) symmetries are related via CKM mixing of the three SU(3) eigenstates in the enlarged unified group GL(4) via \mathbf{U}_{CKM} .

Furthermore, the independent sets of hermitian generators $\{\Gamma^0, \mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3, \tilde{\mathbf{t}}_1, \dots, \tilde{\mathbf{t}}_8, \Delta_1, \Delta_2, \Delta_3\}$ and $\{\Gamma^0, \mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3, \mathbf{T}_1, \dots, \mathbf{T}_{11}\}$ include local *gauge* invariance transformations under SU(3) and SU(2) for *interacting* fields satisfying (1.1) that connect the two basis sets via CKM mixing, where

$$\mathbf{U}^{(3)}(\underline{\alpha}) = e^{i \sum_b \alpha^b \tilde{\mathbf{t}}_b} \text{ and } \mathbf{U}^{(2)}(\underline{\theta}) = e^{i \sum_s \theta^s \tau_s}. \quad (3.20)$$

In these expressions, $\underline{\alpha} = \underline{\alpha}(x)$ are local group parameters of SU(3) and $\underline{\theta} = \underline{\theta}(x)$ are local group parameters of SU(2) under gauge transformations of the field $\Psi(x) \rightarrow \tilde{\Psi}(x) = \mathbf{U}^{(3,2)}(\underline{\beta}(x))\Psi(x)$. The topology of the local mapping of gauge group parameters $\underline{\beta}(x)$ in space-time determines the monopole structure of the sources of the gauge interactions (see section 4.2 of [6]), as well as the properties of any higher dimensional structures (e.g., strings, etc.) in the formulation.

4 Symmetric Tensors and Conservation Properties

It is of interest to determine whether a symmetric energy-momentum tensor suitable for geometrodynamics can be generated from a Lagrangian that produces the spinor field equations. It should be noted that

the equations that define the properties of the spinors given in (1.1) generally need not be the same as the equations used to construct the energy momentum tensor, and one can *always* construct symmetric energy momentum tensors from a Lagrangian of the form

$$\mathcal{L} = \frac{1}{2} \left\{ \frac{1}{m} \left[\left(\frac{\hbar}{i} \frac{\partial}{\partial x^\mu} - \frac{q_a}{c} \sum_s A_\mu^s \mathbf{G}_s \right) \Psi \right]^\dagger g^{\mu\nu} \left(\frac{\hbar}{i} \frac{\partial}{\partial x^\nu} - \frac{q_a}{c} \sum_s A_\nu^s \mathbf{G}_s \right) \Psi + mc^2 \Psi^\dagger \Psi \right\} + \left. -\frac{1}{16\pi} \text{Tr}(\mathbf{F}^{\mu\nu} \mathbf{F}_{\mu\nu}) - \frac{c^4}{16\pi G_N} \mathcal{R}, \right. \quad (4.21)$$

where g is the determinant of the space-time metric tensor, $\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu - i \frac{q_a}{\hbar c} [\mathbf{A}_\mu, \mathbf{A}_\nu]$, $\mathbf{A}_\mu \equiv \sum_s A_\mu^s \mathbf{G}_s$, and \mathcal{R} is the Ricci scalar. Gravitational field equations result from (4.21) by requiring that $\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} (\sqrt{-g} \mathcal{L}) = 0$. Causal fields (3.8) that satisfy the spinor field equations (1.1) will *also* satisfy the Klein-Gordon equation resulting from this Lagrangian form, which directly produces symmetric energy-momentum tensors.

To determine if the symmetric, locally conserved tensors can also be constructed from a Lagrangian form that generates the spinor field equations, it suffices to examine non-interacting fields. Spinor field equations (1.1) will result from a Lagrangian of the form

$$\mathcal{L}_s = \frac{1}{2} \bar{\Psi} \left[\Gamma^\mu \cdot \left(\frac{\hbar}{i} \frac{\partial}{\partial \xi^\mu} \right) \Psi + \gamma mc \Psi \right] + cc = \frac{1}{2} \bar{\Psi} \left[\Gamma^\mu \frac{\partial x^\beta}{\partial \xi^\mu} \cdot \left(\frac{\hbar}{i} \frac{\partial}{\partial x^\beta} \right) \Psi + \gamma mc \Psi \right] + cc. \quad (4.22)$$

where $\bar{\Psi}$ is the Dirac conjugate of field Ψ , cc refers to the complex conjugate of the previous term, and ξ^μ are locally flat coordinates. A conserved second rank tensor can be constructed consistent with the Euler-Lagrange equations of the form

$$\Pi^\mu \equiv \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Psi)}, \quad \mathcal{T}^{\mu\nu} = \Pi^\mu g^{\nu\beta} \partial_\beta \Psi + cc - g^{\mu\nu} \mathcal{L}, \quad (4.23)$$

However, in this case the tensor $\mathcal{T}^{\mu\nu}$ is not symmetric. Following Belinfante[12][13], the tensor describing internal transformations under general Lorentz transformations given by

$$\mathcal{S}^{\beta\mu\nu} = \frac{i\hbar}{2} (\Pi^\beta [\Gamma^\mu, \Gamma^\nu] \Psi - \Pi^\mu [\Gamma^\beta, \Gamma^\nu] \Psi - \Pi^\nu [\Gamma^\beta, \Gamma^\mu] \Psi) + cc \quad (4.24)$$

can be used to construct a symmetric second rank tensor $T^{\nu\mu} = T^{\mu\nu} \equiv \mathcal{T}^{\mu\nu} + \partial_\beta \mathcal{S}^{\beta\mu\nu}$ that satisfies the conservation equation $\partial_\mu T^{\mu\nu} = 0$ in flat space-time (since $\mathcal{S}^{\beta\mu\nu}$ is antisymmetric under the interchange $\beta \leftrightarrow \mu$). However, using curvilinear coordinates, the results of the construction are as follows:

$$T^{\mu\nu} \equiv \mathcal{T}^{\mu\nu} + D_\beta \mathcal{S}^{\beta\mu\nu}, \quad D_\mu T^{\mu\nu} = -\frac{1}{2} R^\nu_{\lambda\beta\mu} \mathcal{S}^{\beta\mu\lambda}. \quad (4.25)$$

Thus, the covariant divergence of the Belinfante construct does not generally vanish in curved space-times. That this construct does not generate an energy-momentum tensor that can drive the geometrodynamics should not be surprising, since for integer values of Γ , there are states satisfying (1.1) with finite mass, yet vanishing eigenvalues of $\Gamma^\mu \hat{P}_\mu$, for which $\gamma = 0$. These states must nonetheless carry non-vanishing energy-momentum.

5 Conclusions

The fundamental representation of causal spinor fields has been shown to unify a set of internal local symmetries including a $U(1)$ symmetry along with 11 additional hermitian generators that can represent a linearly independent $SU(2)$ symmetry or a linearly independent $SU(3)$ symmetry, but not both simultaneously. The eigenbasis of the $SU(3)$ internal symmetry has been related to that of the $SU(2)$ internal symmetry via a CKM transformation mixing the symmetries in $GL(4)$ consistent with observed phenomenology. The closed representation of $SU(2)$ can share the same basis as that in which Γ^0 and J_3 are diagonal, whereas the $SU(3)$ eigenbasis with two diagonal generators cannot share this basis.

Group algebraic invariants *explicitly* generate the Minkowski metric using the non-abelian algebra of generators that carry single space-time indexes, thereby extending interior group structure to the dynamics of general coordinate transformations. The geometrodynamics of general relativity follows directly via the principle of equivalence.

On-going efforts examine how the gauge bosons of the internal $SU(2) \times U(1)$ pre-symmetry of the $\Gamma = \frac{1}{2}$ representation can relate to the quanta of the *massive* $\Gamma = 1$ boson representation with $N_{\Gamma=1} = 10$, which contains a self-adjoint scalar particle, a self-adjoint vector particle, another vector particle, and its adjoint vector particle. The self-adjoint $\gamma = 0$ states are degenerate in their eigenvalue, and can freely mix massive as well as massless vector states in a manner that preserves the degrees of freedom.

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